

Supplemental Notes

EES03 week 01

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Probability for Electrical and Computer Engineers.

HW #1

① HW #1 handout

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Topics

- Truth table proofs
- IMAC (format for exam)
- Probability Spaces

CUT CAT

- Set functions

$$f(A) = \{ f(x) \in Y : x \in A \}.$$

"pull-back"

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}$$

Theorem (1) $P \wedge Q \Rightarrow P$

(2) $P \Rightarrow P \vee Q$

Prf: by (binary) truth table.

P	Q	P	Q	Q	$\Rightarrow$	P	P	$\Rightarrow$	P	$\vee$	Q
1	1	1	1	1	1	1	1	1	1	1	1
0	1	0	0	1	1	0	0	1	0	1	1
1	0	1	0	0	1	1	1	1	1	1	0
0	0	0	0	0	1	0	0	1	0	0	0

QED.

IMAC format

I: Issue

- associative memory ("spot the issue")

M: Math rule

- rote memorization

A: Apply math to facts (Analysis)

- pattern matching
- element because fact.

C: Conclusion

- deductive logic

IMAC example: finite sigma-algebra

Sample space X contains four elements $X = \{a, b, c, d\}$.
So its power set or set of subsets 2^X contains 2^4 elements. Thus the power set 2^X gives rise to 2^{16} subcollections of sets. Suppose that $\mathcal{A} \subset 2^X$ is one of these 2^{16} set subcollections of 2^X and that $\mathcal{A}$ contains the following 7 sets:

$$\mathcal{A} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}, X\}.$$

Can we define a probability space $(X, \mathcal{A}, P)$ using the set collection $\mathcal{A}$ if P is a probability measure?

(use IMAC format to answer)

Soln:

I: sigma-algebra (s.a.)

M: $\mathcal{Q} \subset 2^X$ is a sigma-algebra

iff $\mathcal{Q}$ is CUT:

$$\left\{ \begin{array}{l} C: A \in \mathcal{Q} \implies A^c \in \mathcal{Q} \\ U: A_1 \in \mathcal{Q}, A_2 \in \mathcal{Q}, \dots \implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{Q} \\ T: X \in \mathcal{Q} \end{array} \right.$$

A: $\{a\}^c = \{b, c, d\} \in \mathcal{Q}$

C: $\{b\}^c = \{a, c, d\} \in \mathcal{Q}$.

$\{a, b\}^c = \{c, d\} \notin \mathcal{Q}$

$$\left(\begin{array}{l} \{b, c, d\}^c = \{a\} \in \mathcal{Q} \\ \{a, c, d\}^c = \{b\} \in \mathcal{Q} \\ \emptyset^c = X \in \mathcal{Q}, X^c = \emptyset \in \mathcal{Q} \end{array} \right)$$

$\therefore$ $\mathcal{A}$ is not closed under complement

because $\{a, b\}^c \notin \mathcal{A}$

u: $\{a\} \cup \{b\} = \{a, b\} \in \mathcal{A}$

$$\{a\} \cup \{a, b\} = \{a, b\} \in \mathcal{A}$$

$$\{a\} \cup \{a, c, d\} = \{a, c, d\} \in \mathcal{A}$$

$$\{a\} \cup \{b, c, d\} = X \in \mathcal{A}$$

$$\{b\} \cup \{a, b\} = \{a, b\} \in \mathcal{A}$$

$$\{b\} \cup \{a, c, d\} = X \in \mathcal{A}$$

$$\{b\} \cup \{b, c, d\} = \{b, c, d\} \in \mathcal{A}$$

$$\emptyset \cup S = S \in \mathcal{A} \quad \text{if } S \in \mathcal{A}$$

$$X \cup S = X \in \mathcal{A} \quad \text{if } S \in \mathcal{A}$$

$\therefore$ $\mathcal{A}$ is closed under union.

T: $X \in \mathcal{A}$.

$\therefore \mathcal{A}$ is ~~σ~~ U T

$\therefore \mathcal{A}$ is not a sigma-algebra.

C: **NO.**

$(X, \mathcal{A}, P)$ cannot be a probability space
because $\mathcal{A}$ is not a sigma-algebra.

CUT CAT

- acronym (mnemonic) for a probability space

① CUT

- definition of a sigma-algebra

Space X ("Sample space" Ω)

$2^X = \{A: A \subset X\}$ - power set
- set of all subsets.

$$\therefore \underline{x \in A \subset X \in \mathcal{Q} \subset 2^X.}$$

if $\mathcal{Q}$ is a subcollection of 2^X (that contains X)
↑
set of sets

Defn: $\mathcal{Q} \subset 2^X$ is a sigma-algebra (s.a. or σ -algebra)
iff $\mathcal{Q}$ is CUT:

C: closed under complement.

$$A \in \mathcal{Q} \implies A^c \in \mathcal{Q}$$

U: closed under countable unions

$$A_1 \in \mathcal{Q}, A_2 \in \mathcal{Q}, \dots \implies \bigcup_k A_k \in \mathcal{Q}$$

T: total set X

$$X \in \mathcal{Q}$$

Defn: A is an event (is "measurable") iff
 $A \in \mathcal{Q}$ and $\mathcal{Q}$ is a S.A.

Defn: $(X, \mathcal{Q})$ is a measurable space iff
 $\mathcal{Q} \subset 2^X$ is a S.A.

② CAT

- definition of a probability measure

Defn: Suppose $(X, \mathcal{Q})$ is a measurable space.

Then $P: \mathcal{Q} \rightarrow [0, 1]$ is a probability measure iff P is CAT.

CA: Countable Additive

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k) \quad \text{if } A_i \cap A_j = \emptyset \text{ when } i \neq j$$

(A_i and A_j are "mutually exclusive")

T: Total set X gets unit mass:

$$P(X) = 1$$

Addition Theorem

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Set functions

points to points :

$$f: X \longrightarrow Y$$

$$x \longmapsto f(x)$$

sets to sets

$$f: 2^X \longrightarrow 2^Y$$

$$f^{-1}: 2^Y \longrightarrow 2^X$$

Defn: Set image $f(A) = \{f(x) : x \in A\}$.

★ Defn: Inverse image ("pullback")

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Note: set inverse $f^{-1}: 2^Y \longrightarrow 2^X$ always exists.

Defn: $(X, \mathcal{A}, P)$ is a probability space iff

$\mathcal{A} \subset 2^X$ is CAT and

$P: \mathcal{A} \longrightarrow [0, 1]$ is CAT

$\therefore$ CAT CAT

Thrm. $f(A \cup B) \subset f(A) \cup f(B)$

Prf: pick $y = f(x) \in f(A \cup B)$
($\exists x \in X: y = f(x)$ and $x \in A \cup B$)

$$\therefore x \in A \cup B$$

$$\therefore x \in A \quad \text{OR} \quad x \in B$$

$$\therefore f(x) \in f(A) \quad \text{OR} \quad f(x) \in f(B).$$

$$\therefore f(x) \in f(A) \cup f(B).$$

$$\therefore f(A \cup B) \subset f(A) \cup f(B)$$

Assume

def f

def $\cup$

def f

def $\cup$

def $\subset$

QED.

Thrm: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Prf:

Claim 1: $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Prf:

$$\text{pick } x \in f^{-1}(A \cup B)$$

$$\therefore f(x) \in A \cup B$$

$$\therefore f(x) \in A \quad \text{OR} \quad f(x) \in B.$$

$$\therefore x \in f^{-1}(A) \quad \text{OR} \quad x \in f^{-1}(B)$$

$$\therefore x \in f^{-1}(A) \cup f^{-1}(B)$$

$$\therefore f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$$

Assume

def f^{-1}

def $\cup$

def f^{-1}

def $\cup$

def $\subset$

QED (claim 1)

Claim 2: $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Prf:

pick $x \in f^{-1}(A) \cup f^{-1}(B)$

$\therefore x \in f^{-1}(A) \quad \text{OR} \quad x \in f^{-1}(B)$

$\therefore f(x) \in A \quad \text{OR} \quad f(x) \in B$

$\therefore f(x) \in A \cup B$

$\therefore x \in f^{-1}(A \cup B)$

$\therefore f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Assume

def $\cup$

def f^{-1}

def $\cup$

def f^{-1}

def $\subset$

QED (Claim 2)

$\therefore \text{Claim 1 \& Claim 2} \implies$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

def =

QED

Thm: (Distributivity)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Prf:

Lemma: $P \& (Q \vee R) = (P \& Q) \vee (P \& R)$

Prf: by truth table, propositional calculus

P	Q	R	$P \& (Q \vee R) = (P \& Q) \vee (P \& R)$									
1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1	1	0	0	1
0	0	1	0	0	0	1	1	1	0	0	0	1
1	1	0	1	1	1	1	0	1	1	1	0	0
0	1	0	0	0	1	1	0	1	0	0	0	0
1	0	0	1	0	0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0

$\therefore$ Valid ("tautology")

$\therefore$ QED (lemma)

Claim 1: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Prf: pick $x \in A \cap (B \cup C)$

Assume

$$\therefore x \in A \text{ AND } x \in (B \cup C)$$

def $\cap$

$$\therefore \underbrace{x \in A}_P \text{ AND } \left(\underbrace{x \in B}_Q \text{ OR } \underbrace{x \in C}_R \right) \text{ for fixed } x$$

def $\cup$

$$\therefore (x \in A \& x \in B) \vee (x \in A \& x \in C)$$

lemma

$$\therefore (x \in A \cap B) \vee (x \in A \cap C)$$

defn $\cap$

$$\therefore x \in (A \cap B) \cup (A \cap C)$$

defn $\cup$

$$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

defn $\subseteq$

QED (claim 1)

Claim 2: $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

Prf:

pick $x \in (A \cap B) \cup (A \cap C)$

Assume

$$\therefore (x \in A \cap B) \vee (x \in A \cap C)$$

def $\cup$

$$\therefore (x \in A \ \& \ x \in B) \vee (x \in A \ \& \ x \in C)$$

def $\cap$

$$\therefore x \in A \ \& \ (x \in B \vee x \in C)$$

lemma

$$\therefore x \in A \ \& \ x \in (B \cup C)$$

def $\cup$

$$\therefore x \in A \cap (B \cup C)$$

def $\cap$

$$\therefore (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

def $\subseteq$

QED (claim 2)

$\therefore$ Claim 1 and Claim 2 $\implies$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

def =

QED.

Thrm: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(similar lemma and proof)